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A TECHNIQUE FOR ANALYSING FINITE ELEMENT METHODS FOR VISCOUS INCOMPRESSIBLE FLOW

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SUMMARY

We give a self-contained presentation of our macroelement technique for verifying the stability of finite element discretizations of the Navier–Stokes equations in the velocity–pressure formulation.

KEY WORDS Stokes equations Mixed Finite elements Stability Patch test

1. INTRODUCTION

In this paper we will consider some aspects of the approximate solution of the incompressible Navier–Stokes equations by finite element methods. The class of methods to be discussed consists of methods where independent finite element spaces are used for the velocity and the pressure.

There are two difficult problems connected with this approach. The first is the approximation of the convection term. Recently, considerable progress has been made with this problem, as can be seen from some of the other papers in this issue.

The problem we are going to discuss stems from the incompressibility condition. It is well known that this implies that the finite element spaces for the velocity and the pressure cannot be chosen arbitrarily. Instead, the velocity–pressure pair has to satisfy a stability inequality, the famous 'Babuška–Brezzi' or 'inf–sup' condition.

The basic theory for mixed methods was developed in the fundamental papers by Babuška[1,2] and Brezzi.[3] Later, this theory was applied to mixed finite element methods for a number of problems in continuum mechanics.

With regards to the discretization of the Navier–Stokes equations, a technique for proving the Babuška–Brezzi condition was introduced in a basic paper by Crouzeix and Raviart.[4] In the same paper a widely used technique for designing stable discretizations using 'bubble functions' was introduced.

The drawback of the technique of Reference 4 is that it consists of an explicit construction of the stability inequality, and this involves some technical scaling arguments. In particular, the technique is difficult to apply to so-called Taylor–Hood methods in which continuous approximations are used for the pressure.

The problem of analysing Taylor–Hood methods was partially resolved by Bercovier and Pironneau,[5] who showed that the convergence can be proved by altering the norms used in the stability inequality. Later, Verfürth[6] considerably simplified their analysis by showing that the modified stability inequality implies the inequality with the natural norms.

We refer to the book by Girault and Raviart[7] for a rather complete survey of methods which have been proven to be stable.

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The purpose of this paper is to give a self-contained review of a technique developed by us[8,9] for the analysis of mixed methods (see also Reference 10 for very related ideas). Our results show that the Babuška–Brezzi inequality can be proved by verifying similar local inequalities posed over 'macroelements' consisting of a finite number of elements. Furthermore, these local stability estimates are equivalent to a simple algebraic condition which often can easily be checked. As a result, many technical arguments previously needed when analysing a method can be avoided. In engineering language our technique consists of a 'patch test' that has to be verified. Another related but non-rigorous (see Remark 2 below) 'patch test' has recently been advocated.[11]

The plan of the paper is the following. In the next section we briefly recall some background results and definitions. Section 3 is devoted to our analysis technique, which is applied to some examples in Section 4.

We only treat conforming methods, but the technique can also be applied to non-conforming approximations. This has recently been done in Reference 12.

We would like to emphasize that the results of the paper also cover the analysis of mixed methods for (nearly) incompressible elasticity.

Let us also point out that the same technique can be applied for the analysis of mixed finite element methods for other problems, e.g. the equations of elasticity with the displacement and the stress tensor as independent variables.

2. PRELIMINARIES

Since we are not concerned with the discretization of the convection term in the Navier–Stokes equations, it will be sufficient to consider the approximation of the Stokes equations with viscosity equal to unity: find the velocity $\mathbf{u} = (u_1, \ldots, u_d)$ and the pressure p such that

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where, as usual, non-homogeneous boundary conditions are included in the body force vector $\mathbf{f}$. The domain $\Omega \subset \mathbf{R}^d$, $d=2$ or 3, is assumed to be bounded and, for simplicity, polygonal or polyhedral.

The mathematical formulation of the problem is: find $\mathbf{u} \in [H_0^1(\Omega)]^d$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} (\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla\cdot\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in [H_0^1(\Omega)]^d, \\ (\nabla\cdot\mathbf{u}, q) &= 0, \quad q \in L_0^2(\Omega). \end{aligned} \tag{2}$$

Our notation is standard.[13] $[H^s(D)]^\alpha$, with $\alpha = 1$ or d and integer s, denotes the standard $L^2(D)$-based Sobolev spaces. $L_0^2(D)$ denotes the subspace of $L^2(D)$ of functions with zero mean value:

$$L_0^2(D) = \left\{ p \in L^2(D) \,\middle|\, \int_D p\,\mathrm{d}x = 0 \right\}.$$

The norms and seminorms in $[H^s(D)]^\alpha$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ respectively. Furthermore, $(\,.\,,\,.\,)_D$ denotes the inner product in $L^2(D)$, $[L^2(D)]^d$ or $[L^2(D)]^{d\times d}$. As usual, the subscripts are dropped for the case $D = \Omega$. By C and C_j, $j \in \mathbf{N}$, we denote various positive constants which do not necessarily take the same values at each instance. Furthermore, these constants are independent of the element and macroelement partitionings $\mathscr{C}_h$ and $\mathscr{M}_h$ to be introduced.

The problem (2) is a typical example of a saddle-point problem, and the existence and uniqueness of the solution are a consequence of the inequality

$$\sup_{0 \neq \mathbf{v} \in [H_0^1(\Omega)]^d} \frac{(\nabla \cdot \mathbf{v}, p)}{\|\mathbf{v}\|_1} \geqslant C\|p\|_0, \quad p \in L_0^2(\Omega). \tag{3}$$

A simple proof of this in the case when Ω has a smooth boundary can be found in Reference 14, pp. 172–174. For the general case we refer to Reference 7.

The class of methods we are concerned with is formulated as follows. We choose two subspaces $\mathbf{V}_h \subset [H_0^1(\Omega)]^d$ and $P_h \subset L_0^2(\Omega)$ and pose the problem: find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$\begin{aligned} (\nabla \mathbf{u}_h, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p_h) &= (\mathbf{f}, \mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h, \\ (\nabla \cdot \mathbf{u}_h, q) &= 0, \quad q \in P_h. \end{aligned} \tag{4}$$

Now, in order to have a good finite element method, the finite element spaces have to be chosen so that they inherit the property (3), i.e. they should satisfy

$$\sup_{0 \neq \mathbf{v} \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v}, p)}{\|\mathbf{v}\|_1} \geqslant C\|p\|_0, \quad p \in P_h. \tag{5}$$

Then the theory of mixed methods states that the following optimal error estimate is valid:

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant C\left\{\inf_{\mathbf{v} \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}\|_1 + \inf_{q \in P_h} \|p - q\|_0\right\}. \tag{6}$$

3. THE ANALYSIS TECHNIQUE

In order to be able to give precise and general results, we have to define our concepts properly. This will unfortunately burden the presentation, but the main idea should, however, not be difficult to grasp.

The reader mainly interested in applications of finite element methods may skip the technical details in the proofs of the stability and error estimates, and instead go to Theorem 2 after getting acquainted with our definitions.

We let $\mathscr{C}_h$ be a partitioning of $\bar{\Omega}$ into elements which are all assumed to be either triangles or convex quadrilaterals in the two-dimensional case, and tetrahedrons or convex hexahedrons for a three-dimensional problem. Naturally, the partitioning is assumed to satisfy the usual compatibility and regularity conditions.[13] As an example we recall the definition of regularity for a triangular partitioning. Given an element $K \in \mathscr{C}_h$, let h_K denote the diameter of K and let ρ_K be the maximum diameter of all circles contained in K. $\mathscr{C}_h$ is then regular if there is a constant $\sigma > 1$ such that

$$h_K \leqslant \sigma \rho_K \quad \text{for all} \quad K \in \mathscr{C}_h. \tag{7}$$

For the other type of elements the regularity is defined analogously.[13]

Let us further assume that the finite element spaces can be uniquely defined using a reference element $\hat{K}$ (i.e. the unit triangle, tetrahedron, square or cube) and two finite-dimensional polynomial spaces $\hat{\mathbf{V}}$ and $\hat{P}$ defined on $\hat{K}$. For $K \in \mathscr{C}_h$ we let F_K be the affine, bilinear or trilinear mapping from $\hat{K}$ onto K. We then define

$$\mathbf{V}_h = \{\mathbf{v} \in [H_0^1(\Omega)]^d \,|\, \mathbf{v}(x) = \hat{\mathbf{v}}(F_K^{-1}(x)), \quad \hat{\mathbf{v}} \in \hat{\mathbf{V}}, \quad K \in \mathscr{C}_h\} \tag{8}$$

and

$$P_h = \{p \in L_0^2(\Omega) \,|\, p(x) = \hat{p}(F_K^{-1}(x)), \quad \hat{p} \in \hat{P}, \quad K \in \mathscr{C}_h\} \tag{9a}$$

or

$$P_h=\{p\in C(\Omega)\cap L_0^2(\Omega)\,|\,p(x)=\hat{p}(F_K^{-1}(x)),\quad \hat{p}\in\hat{P},\quad K\in\mathscr{C}_h\}. \tag{9b}$$

The choice (9b) gives a method of the Taylor–Hood type.

Next let us introduce the concept of a macroelement, i.e. a connected set which is the union of at least two elements. For the elements of a macroelement we also impose the usual compatibility and regularity conditions.

Given a macroelement M, we define finite element spaces consistent with (8) and (9):

$$\mathbf{V}_{0,M}=\{\mathbf{v}\in[H_0^1(M)]^d\,|\,\mathbf{v}(x)=\hat{\mathbf{v}}(F_K^{-1}(x)),\quad \hat{\mathbf{v}}\in\hat{\mathbf{V}},\quad x\in K,\quad K\subset M\} \tag{10}$$

and

$$P_M=\{p\in L^2(M)\,|\,p(x)=\hat{p}(F_K^{-1}(x)),\quad \hat{p}\in\hat{P},\quad x\in K,\quad K\subset M\} \tag{11a}$$

or

$$P_M=\{p\in C(M)\,|\,p(x)=\hat{p}(F_K^{-1}(x)),\quad \hat{p}\in\hat{P},\quad x\in K,\quad K\subset M\} \tag{11b}$$

By Γ_h we denote the collection of edges or faces (for a three-dimensional problem), of the elements of $\mathscr{C}_h$, not lying on the boundary of Ω.

The following norm defined in P_h turns out to be very useful:

$$\|p\|_h^2=\sum_{K\in\mathscr{C}_h}h_K^2\|\nabla p\|_{0,K}^2+\sum_{T\in\Gamma_h}h_T\int_T|[\![p]\!]|^2\,\mathrm{d}s.$$

Here and in the sequel T stands for an edge or a face of an element and h_T denotes the diameter of T. $([\![p]\!])_{|T}$ denotes the jump in p along T.

In P_M we similarly define

$$|p|_M^2=\sum_{K\subset M}h_K^2\|\nabla p\|_{0,K}^2+\sum_{T\in\Gamma_M}h_T\int_T|[\![p]\!]|^2\,\mathrm{d}s,$$

where Γ_M denotes the interior edges (faces) of M, i.e. $\Gamma_M=\{T\subset M\,|\,T\not\subset\partial M\}$.

The usefulness of the macroelement concept and the above mesh-dependent norms is that it enables us to build a global stability estimate by simply adding together analogous local estimates.

Lemma 1

Suppose that we can define a macroelement partitioning $\mathscr{M}_h$ such that:

(i) Each $T\in\Gamma_h$ is an interior edge (face) of at least one and not more than L macroelements of $\mathscr{M}_h$.

(ii) There is a positive constant C such that

$$\sup_{0\neq\mathbf{v}\in\mathbf{V}_{0,M}}\frac{(\nabla\cdot\mathbf{v},p)_M}{|\mathbf{v}|_{1,M}}\geqslant C|p|_M,\quad p\in P_M, \tag{12}$$

holds for all $M\in\mathscr{M}_h$.

Then the stability inequality

$$\sup_{0\neq\mathbf{v}\in\mathbf{V}_h}\frac{(\nabla\cdot\mathbf{v},p)}{\|\mathbf{v}\|_1}\geqslant C\|p\|_h,\quad p\in P_h. \tag{13}$$

is valid.

Proof. Let $p \in P_h$ be arbitrary. The local stability estimates imply that for every $M \in \mathcal{M}_h$ there exists $\mathbf{v}_M \in \mathbf{V}_h$, vanishing outside of M, such that

$$(\nabla \cdot \mathbf{v}_M, p) = (\nabla \cdot \mathbf{v}_M, p)_M \geqslant C|p|_M^2$$

and

$$|\mathbf{v}_M|_1 = |\mathbf{v}_M|_{1,M} \leqslant |p|_M.$$

Let us define

$$\mathbf{v} = \sum_{M \in \mathcal{M}_h} \mathbf{v}_M.$$

Since each $T \in \Gamma_h$ is an interior edge (face) of at least one $M \in \mathcal{M}_h$, each element $K \in \mathscr{C}_h$ is contained in one macroelement of $\mathcal{M}_h$. Hence we have

$$(\nabla \cdot \mathbf{v}, p) = \sum_{M \in \mathcal{M}_h} (\nabla \cdot \mathbf{v}_M, p) \geqslant C \sum_{M \in \mathcal{M}_h} |p|_M^2 \geqslant C \|p\|_h^2.$$

Furthermore, since each $T \in \Gamma_h$ is contained in at most L macroelements, each element $K \in \mathscr{C}_h$ is contained in at most $6L$ macroelements (with the maximum obtained for a hexahedral partitioning). This gives

$$\|\mathbf{v}\|_1 \leqslant C|\mathbf{v}|_1 \leqslant C \sum_{M \in \mathcal{M}_h} |\mathbf{v}_M|_1 \leqslant C \sum_{M \in \mathcal{M}_h} |p|_M \leqslant 6CL\|p\|_h,$$

which together with the earlier estimate prove the assertion. □

The following two results, essentially due to Verfürth,[6] provide the link between the stability in the mesh-dependent norm and in the desired L^2-norm. We formulate them as separate lemmas, since the same reasoning can be used in other contexts as well.[15,16] We remark that the lemma below is valid for arbitrary spaces $\mathbf{V}_h$ and P_h.

Lemma 2

There are two positive constants C_1, C_2 such that

$$\sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{v}, p)}{\|\mathbf{v}\|_1} \geqslant C_1 \|p\|_0 - C_2 \|p\|_h, \quad p \in P_h.$$

Proof. Let $p \in P_h$ be arbitrary. The condition (3) then implies the existence of $\mathbf{w} \in [H_0^1(\Omega)]^d$ such that

$$(\nabla \cdot \mathbf{w}, p) \geqslant C_3 \|p\|_0^2 \tag{14}$$

and

$$\|\mathbf{w}\|_1 \leqslant \|p\|_0. \tag{15}$$

We now interpolate $\mathbf{w}$ with $\tilde{\mathbf{w}} \in \mathbf{V}_h$ defined by the technique of Clemént,[17] so that we have the estimates (see Reference 18, Lemma 3 and Reference 7, pp. 109–111)

$$\left(\sum_{K \in \mathscr{C}_h} h_K^{-2} \|\mathbf{w} - \tilde{\mathbf{w}}\|_{0,K}^2 + \sum_{T \in \Gamma_h} h_T^{-1} \int_T |\mathbf{w} - \tilde{\mathbf{w}}|^2 \, ds \right)^{1/2} \leqslant C_4 |\mathbf{w}|_1 \tag{16}$$

and

$$\|\tilde{\mathbf{w}}\|_1 \leqslant C_5 \|\mathbf{w}\|_1. \tag{17}$$

Integrating by parts on each $K\in\mathscr{C}_h$ and using (14) and (16), we get

$$\begin{aligned}(\nabla\cdot\tilde{\mathbf{w}},p)&=(\nabla\cdot(\tilde{\mathbf{w}}-\mathbf{w}),p)+(\nabla\cdot\mathbf{w},p)\\&\geqslant(\nabla\cdot(\tilde{\mathbf{w}}-\mathbf{w}),p)+C_3\|p\|_0^2\\&=\sum_{K\in\mathscr{C}_h}(\mathbf{w}-\tilde{\mathbf{w}},\nabla p)_K+\sum_{T\in\Gamma_h}\int_T((\tilde{\mathbf{w}}-\mathbf{w})\cdot\mathbf{n})([\![p]\!])\,\mathrm{d}s+C_3\|p\|_0^2\\&\geqslant-\left(\sum_{K\in\mathscr{C}_h}h_K^{-2}\|\mathbf{w}-\tilde{\mathbf{w}}\|_{0,K}^2+\sum_{T\in\Gamma_h}h_T^{-1}\int_T|\mathbf{w}-\tilde{\mathbf{w}}|^2\,\mathrm{d}s\right)^{1/2}\|p\|_h+C_3\|p\|_0^2\\&\geqslant-C_4|\mathbf{w}|_1\|p\|_h+C_3\|p\|_0^2\\&\geqslant-C_4\|p\|_0\|p\|_h+C_3\|p\|_0^2\\&=(C_3\|p\|_0-C_4\|p\|_h)\|p\|_0.\end{aligned}\tag{18}$$

Equations (17) and (15) now give

$$\|\tilde{\mathbf{w}}\|_1\leqslant C_5\|\mathbf{w}\|_1\leqslant C_5\|p\|_0.\tag{19}$$

Hence (18) and (19) give the asserted estimate. □

Lemma 3

Suppose that the stability estimate (13) is valid. Then the desired stability condition (5) also holds.

Proof. Let C_1, C_2 be the constants in Lemma 2 and denote by C_3 that of (13). For $0<\xi<1$ we then have

$$\begin{aligned}\sup_{0\neq\mathbf{v}\in\mathbf{V}_h}\frac{(\nabla\cdot\mathbf{v},p)}{\|\mathbf{v}\|_1}&=\xi\sup_{0\neq\mathbf{v}\in\mathbf{V}_h}\frac{(\nabla\cdot\mathbf{v},p)}{\|\mathbf{v}\|_1}+(1-\xi)\sup_{0\neq\mathbf{v}\in\mathbf{V}_h}\frac{(\nabla\cdot\mathbf{v},p)}{\|\mathbf{v}\|_1}\\&\geqslant C_1\xi\|p\|_0+[(1-\xi)C_3-\xi C_2]\|p\|_h\\&\geqslant C\|p\|_0\end{aligned}$$

when choosing $\xi<C_3(C_2+C_3)^{-1}$. □

By Lemmas 1–3, the problem of proving the stability inequality is reduced to proving the local estimates (12).

An immediate observation is that a necessary condition for the inequality (12) to be valid is that the subspace

$$N_M=\{p\in P_M\,|\,(\nabla\cdot\mathbf{v},p)_M=0,\quad\mathbf{v}\in\mathbf{V}_{0,M}\}$$

only consists of the functions which are constant on M. We call this the 'macroelement condition'.

In Reference 8 we showed, roughly speaking, that if this condition is satisfied independently of the geometrical shape of the macroelement, then the local stability estimate, with a constant independent of the particular macroelement, is valid. For stating the exact result we need one more definition.

A macroelement M is said to be equivalent with a reference macroelement $\hat{M}$ if one can define a continuous one-to-one mapping F_M: $\hat{M}\to M$ such that:

(i) $F_M(\hat{M})=M$.

(ii) If $\hat{M}=\cup_{j=1}^m\hat{K}_j$, where $\hat{K}_j, j=1,2,\ldots,m$, are the elements of $\hat{M}$, then $K_j=F_M(\hat{K}_j)$, $j=1,2,\ldots,m$, are the elements of M.

(iii) $F_{M|\hat{K}_j} = F_{K_j} \circ F_{\hat{K}_j}^{-1}, j = 1, 2, \ldots, m$, where F_{K_j} and $F_{\hat{K}_j}$ are the mappings from the reference element $\hat{K}$ onto K_j and $\hat{K}_j$ respectively.

Lemma 4

Let $\mathscr{E}$ be a class of equivalent macroelements. Suppose that for every $M \in \mathscr{E}$ the space N_M is one-dimensional, consisting of functions constant on M. Then there is a constant C such that

$$\sup_{0 \neq \mathbf{v} \in \mathbf{V}_{0,M}} \frac{(\nabla \cdot \mathbf{v}, p)_M}{|\mathbf{v}|_{1,M}} \geqslant C|p|_M, \quad p \in P_M,$$

holds for all $M \in \mathscr{E}$.

Proof. For $M \in \mathscr{E}$ define

$$\beta_M = \inf_{\substack{p \in P_M \\ |p|_M = 1}} \sup_{\substack{\mathbf{v} \in \mathbf{V}_{0,M} \\ \|\mathbf{v}\|_{1,M} = 1}} (\nabla \cdot \mathbf{v}, p)_M.$$

Since N_M is assumed to consist only of the constant functions, we have $\beta_M > 0$. We thus have to show that there exists a constant $\beta_\varepsilon > 0$ such that $\beta_M \geqslant \beta_\varepsilon > 0$ for every $M \in \mathscr{E}$.

To prove this we use a kind of generalized scaling argument. Denote by $\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^k$ the vertices of $\hat{M}$. Then M is uniquely defined by its vertices $x^i = F_M(\hat{x}^i)$, $i = 1, 2, \ldots, k$. In particular, this means that we can write $\beta_M = \beta(x^1, x^2, \ldots, x^k) = \beta(X)$, with $X = (x^1, x^2, \ldots, x^k)$ considered as a point in $\mathbf{R}^{dk}$. Without loss of generality we can assume that x^1 coincides with the origin, and $h_M = 1$, with $h_M = \max_{K \subset M} h_K$, since the general case can be handled by changing variables from x to $h_M^{-1}(x - x^1)$. By this, every vertex will be within a given distance from the origin. Furthermore, every $K \subset M$ has a diameter less than or equal to unity and satisfies some regularity conditions of the type (7). This means that X belongs to a compact set in $\mathbf{R}^{dk}$. If now the function β can be proven to be continuous, we have

$$\inf_{\substack{p \in P_M \\ |p|_M = 1}} \sup_{\substack{\mathbf{v} \in \mathbf{V}_{0,M} \\ \|\mathbf{v}\|_{1,M} = 1}} (\nabla \cdot \mathbf{v}, p)_M = \beta_M \geqslant \beta_\varepsilon > 0, \quad M \in \mathscr{E},$$

which is equivalent to the asserted estimate.

It is not difficult to see that β is continuous. Let $\tilde{M}$ be another macroelement in $\mathscr{E}$ and denote by $\tilde{X}$ the corresponding point in $\mathbf{R}^{dk}$. Define $G: M \to \tilde{M}$ through $G = F_{\tilde{M}} \circ F_M^{-1}$, where $F_{\tilde{M}}$ and F_M are the mappings from the reference macroelement onto $\tilde{M}$ and M respectively. For arbitrary $\mathbf{v} \in \mathbf{V}_{0,M}$ and $p \in P_M$ we define $\tilde{\mathbf{v}} \in \mathbf{V}_{0,\tilde{M}}$ and $\tilde{p} \in P_{\tilde{M}}$ through

$$\tilde{\mathbf{v}}(\tilde{x}) = \mathbf{v}(G^{-1}(\tilde{x})), \qquad \tilde{p}(\tilde{x}) = p(G^{-1}(\tilde{x})) \qquad \tilde{x} \in \tilde{M}.$$

Let J_G be the Jacobian of G. By transforming integrals posed over $\tilde{M}$ to integrals over M, and using the fact that J_G converges towards the identity when $\tilde{X} \to X$, we now get estimates of the type

$$|(\nabla \cdot \mathbf{v}, p)_M - (\nabla \cdot \tilde{\mathbf{v}}, p)_{\tilde{M}}| \leqslant C_1(X, \tilde{X})|\mathbf{v}|_{1,M}|p|_M,$$

$$||\mathbf{v}|_{1,M} - |\tilde{\mathbf{v}}|_{1,\tilde{M}}| \leqslant C_2(X, \tilde{X})|\mathbf{v}|_{1,M},$$

$$||p|_M - |\tilde{p}|_{\tilde{M}}| \leqslant C_3(X, \tilde{X})|p|_M,$$

with $C_i(X, \tilde{X}) \to 0$, $i = 1, 2, 3$, when $\tilde{X} \to X$.

The continuity of β is now a simple consequence of these three estimates. □

By combining Lemmas 1, 3 and 4 we arrive at our technique for the analysis of mixed methods.

Theorem 1

Suppose that there is a fixed set of equivalence classes $\mathscr{E}_i$, $i = 1, 2, \ldots, l$, of the macroelements, a positive integer L and a macroelement partitioning $\mathscr{M}_h$ such that:

(M1) For each $M \in \mathscr{E}_i$, $i=1, 2, \ldots, l$, the space N_M is one-dimensional, consisting of functions that are constant on M.
(M2) Each $M \in \mathscr{M}_h$ belongs to one of the classes $\mathscr{E}_i$, $i = 1, 2, \ldots, l$.
(M3) Each $T \in \Gamma_h$ is an interior edge (face) of at least one and not more than L macroelements of $\mathscr{M}_h$.

Then the stability inequality (5) is valid.

Proof. Lemma 4 shows that (12), with a constant C_i, holds for each class $\mathscr{E}_i$, $i = 1, 2, \ldots, l$. Letting $C = \min\{C_1, C_2, \ldots, C_l\}$, the assumptions of Lemma 1 are valid and the assertion then follows from Lemma 3. □

Remark 1

Here and in Reference 9 we have chosen to define a macroelement to consist of at least two elements and defined the partitioning $\mathscr{M}_h$ to consist of overlapping macroelements.

In Reference 8 (and also in References 7 and 10) the partitioning is defined such that every element belongs to one and only one macroelement. Furthermore, the local stability estimates, used to build the global one, were

$$\sup_{0 \neq \mathbf{v} \in \mathbf{V}_{0,M}} \frac{(\nabla \cdot \mathbf{v}, p)_M}{|\mathbf{v}|_{1,M}} \geqslant C \|p\|_{0,M}, \quad p \in P_M \cap L_0^2(M).$$

This estimate also follows from the condition that N_M consists of the constant functions.

Since the macroelements were non-overlapping, some additional assumptions had to be made on $\mathscr{M}_h$ in order to assure the stability of the velocity–pressure pair $(\mathbf{V}_h, \bar{P}_h)$, with

$$\bar{P}_h = \{p \in P_h \,|\, p_{|M} \text{ is constant for all } M \in \mathscr{M}_h\}.$$

There are two reasons for introducing the present modification of our technique.

First, by using non-overlapping macroelements, often far more classes are needed, and sometimes it can even be difficult to see how a macroelement partitioning, satisfying all the conditions required, should be constructed. This is particularly true for some three-dimensional methods. A good example is provided by the quadratic tetrahedral Taylor–Hood method. The macroelement condition is easily proven to be satisfied for a macro consisting of tetrahedrons which have exactly one common vertex in the interior of the macroelement.[19] An arbitrary finite element partitioning $\mathscr{C}_h$ cannot, however, be regrouped into non-overlapping macroelements of this type. Also, for those cases for which this would be possible, the condition that the pair $(\mathbf{V}_h, \bar{P}_h)$ be stable is not necessarily valid.

Secondly, by using overlapping macroelements and the local estimates in the form (12), the analysis shows more clearly that it is the condition that N_M consist of the constant functions which is the condition that has to be verified for the macroelements chosen. □

Remark 2

The patch test introduced in Reference 11 consists simply of checking that

$$\dim \mathbf{V}_{0,M} \geqslant \dim P_M - 1.$$

Hence that test is merely the first thing that has to be checked when choosing a candidate for a class satisfying the macroelement condition.

Therefore the patch test of Reference 11 is far from a sufficient condition for the stability of the method, and in fact does not even guarantee that the solution is unique. A good example is the Q9/Q4 element (with the notation of Reference 11), which satisfies the test for a patch of 2×2 elements. However, it is a simple exercise to show that there are meshes for which this method does not yield a unique solution. □

For the practising engineer it is desirable to have a clear understanding of the problem with mixed methods. Let us therefore present the following simplified version of our results.

Theorem 2

Suppose that there is a set of equivalence classes $\mathscr{E}_i$, $i = 1, 2, \ldots, l$, of macroelements and a macroelement partitioning $\mathscr{M}_h$ such that:

(M1) For each $M \in \mathscr{E}_i$, $i = 1, 2, \ldots, l$, the space N_M is one-dimensional, consisting of functions that are constant on M.

(M2) Each $M \in \mathscr{M}_h$ belongs to one of the classes $\mathscr{E}_i$, $i = 1, 2, \ldots, l$.

(M3) Each $T \in \Gamma_h$ is an interior edge (face) of at least one macroelement of $\mathscr{M}_h$.

Then the problem (4) has a unique solution.

Proof. By the linearity we have to show that if

$$(\nabla \mathbf{u}_h, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p_h) = 0, \quad \mathbf{v} \in \mathbf{V}_h,$$

$$(\nabla \cdot \mathbf{u}_h, q) = 0, \quad q \in P_h,$$

then $\mathbf{u}_h = \mathbf{0}$ and $p_h = 0$.

To this end we choose $\mathbf{v} = \mathbf{u}_h$ and $q = p_h$ above. This gives

$$0 = (\nabla \mathbf{u}_h, \nabla \mathbf{u}_h) = \int_\Omega |\nabla \mathbf{u}_h|^2 \,\mathrm{d}x,$$

i.e. $\mathbf{u}_h$ is a constant vector in Ω, and since it vanishes on the boundary of Ω we have $\mathbf{u}_h = \mathbf{0}$.

The first equation above then reduces to

$$(\nabla \cdot \mathbf{v}, p_h) = 0, \quad \mathbf{v} \in \mathbf{V}_h. \tag{20}$$

Now, the conditions (M1)–(M3) ensure that for each interior edge (face) $T \in \Gamma_h$ there is a macroelement M with T in its interior, and a $\mathbf{v}_M \in \mathbf{V}_{0,M} \subset \mathbf{V}_h$ such that choosing $\mathbf{v} = \mathbf{v}_M$ in (20) implies that p_h is constant on M. Since each element has at least one edge (face) in Γ_h, this shows that p_h is constant in the whole of Ω. Owing to the requirement of zero mean value we have $p_h = 0$. □

We see that the proof above is extremely simple and hence could be presented in elementary engineering education. Compared to non-rigorous 'theories' such as the well known 'constraint counting' or the patch test of Reference 11, it has the advantage of giving a simple and completely rigorous condition by which the uniqueness of the approximate solution can be assured.

For the practitioner it should be comforting to know that the condition also implies the strict mathematical stability condition.

4. APPLICATIONS

Let us illustrate our technique by applying it in some concrete examples.

We will first consider a non-standard method introduced by us in Reference 9. The method has at least some pedagogical interest, since it is ideally suited for demonstrating the technique.

Example 1

Let $\mathscr{C}_h$ be a triangulation of the two-dimensional domain. Define

$$\mathbf{V}_h = \{\mathbf{v} = (v_1, v_2) \in [H_0^1(\Omega)]^2 \mid v_{1|K} \in P_1(K), \quad v_{2|K} \in P_2(K), \quad K \in \mathscr{C}_h\},$$

$$P_h = \{p \in L_0^2(\Omega) \mid p_{|K} \in P_0(K), \quad K \in \mathscr{C}_h\}.$$

As macroelements we take the union of elements which all have exactly one common vertex in the interior of the macroelement (see Figure 1). Let us impose the slight restriction on the mesh that every element have at least one vertex in the interior of Ω. $\mathscr{M}_h$ is then constructed by taking for each interior vertex of the mesh one macroelement with this vertex as its interior vertex.

With this the conditions (M2) and (M3) of Theorem 1 are satisfied and it remains to check the condition (M1).

Let M be an arbitrary macroelement of this type and let K_i, $i = 1, 2, \ldots, \kappa$, be the elements of M. The midpoints and the normals to the interior edges of M we denote by x^i and $\mathbf{n}_i$, $i = 1, 2, \ldots, \kappa$, respectively. x^0 is the vertex common to all elements of M. For $p \in P_M$ we let $p_i = p_{|K_i}$, $i = 1, 2, \ldots, \kappa$.

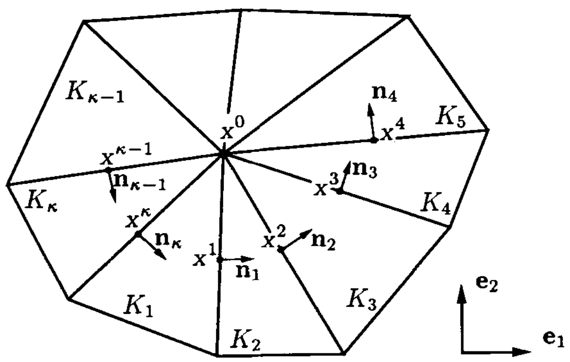

Figure 1

The degrees of freedom for $\mathbf{u} \in \mathbf{V}_{0,M}$ are the values of both components of $\mathbf{u}$ at x^0 and the values $u_2(x^i)$, $i = 1, 2, \ldots, \kappa$. Choosing $\mathbf{u} \in \mathbf{V}_{0,M}$ such that the only non-vanishing degree of freedom is $u_2(x^i)$, the condition $(\nabla \cdot \mathbf{u}, p)_M = 0$ implies that $p_i = p_{i+1}$ (with $p_{\kappa+1} = p_0$) if $\mathbf{n}_i \cdot \mathbf{e}_2 \neq \mathbf{0}$, where $\mathbf{e}_2 = (0, 1)$. Hence the space N_M can be at most two-dimensional, and this happens only if two of the edges are parallel to $\mathbf{e}_2$. However, in this case one chooses $\mathbf{u}$ such that the only non-zero degree of freedom is $u_1(x^0)$. The condition for N_M then forces p to be constant on the whole of M.
The conditions of Theorem 1 are thus valid and hence we get the error estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant Ch(|\mathbf{u}|_2 + |p|_1).$$

The corresponding method can naturally also be defined for a quadrilateral mesh or a mixture of triangles and quadrilaterals. □

In the next examples we will consider the original Taylor–Hood methods.

Example 2[5,6]

We again let $\mathscr{C}_h$ be a triangulation of $\bar{\Omega} \subset \mathbf{R}^2$ and define

$$\mathbf{V}_h = \{\mathbf{v} \in [H_0^1(\Omega)]^2 \mid \mathbf{v}_{|K} \in [P_2(K)]^2, \quad K \in \mathscr{C}_h\},$$

$$P_h = \{p \in C(\Omega) \cap L_0^2(\Omega) \mid p_{|K} \in P_1(K), \quad K \in \mathscr{C}_h\}.$$

For this method the macroelement condition is valid for a macroelement consisting of three elements. To prove this we consider an arbitrary macroelement $M = \cup_{i=1}^{3} K_i$ as shown in Figure 2.

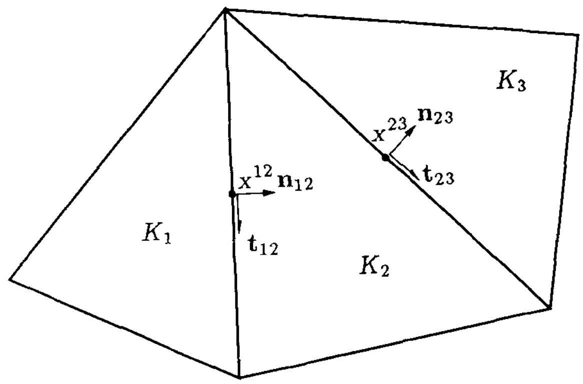

Figure 2

The degrees of freedom for $\mathbf{u} \in \mathbf{V}_{0,M}$ are now the values of $\mathbf{u}$ at the midpoints x^{12} and x^{23} of the edges in the interior of M. $\mathbf{t}_{12}, \mathbf{t}_{23}$ and $\mathbf{n}_{12}, \mathbf{n}_{23}$ denote the tangents and the normals respectively to the interior edges.

Let us choose $\mathbf{u}$ such that $\mathbf{u}(x^{12}) \cdot \mathbf{t}_{12} = 1$, $\mathbf{u}(x^{12}) \cdot \mathbf{n}_{12} = 0$ and $\mathbf{u}(x^{23}) = \mathbf{0}$. Since $\nabla p \cdot \mathbf{t}_{12}$ is constant in $K_1 \cup K_2$, a simple calculation gives

$$(\nabla \cdot \mathbf{u}, p)_M = -(\mathbf{u}, \nabla p)_M = -\tfrac{1}{3}[\text{area}(K_1) + \text{area}(K_2)](\nabla p \cdot \mathbf{t}_{12})_{|K_1 \cup K_2}.$$

Hence, if $p \in N_M$, then

$$\nabla p \cdot \mathbf{t}_{12} = 0 \quad \text{in } K_1 \cup K_2, \tag{21}$$

and by the same argument

$$\nabla p \cdot \mathbf{t}_{23} = 0 \quad \text{in } K_2 \cup K_3. \tag{22}$$

In K_2 we thus have

$$\nabla p = \mathbf{0},$$

i.e. $p \in N_M$ is constant in K_2.

Next we choose $\mathbf{u}$ such that the only non-vanishing degrees of freedom are $\mathbf{u}(x^{12}) \cdot \mathbf{n}_{12}$ and $\mathbf{u}(x^{23}) \cdot \mathbf{n}_{23}$ respectively. The condition for N_M then implies

$$\nabla p \cdot \mathbf{n}_{12} = \mathbf{0} \quad \text{in } K_1$$

and

$$\nabla p \cdot \mathbf{n}_{23} = 0 \quad \text{in } K_3.$$

Together with (21) and (22) this shows that $p \in N_M$ is also constant in K_1 and K_3. Since p is by definition continuous, it is a constant in the whole of M. The macroelement condition is thus satisfied.

The construction of the macroelement partitioning $\mathcal{M}_h$ is now simple; for each interior edge of $\mathscr{C}_h$ we take one macroelement with this edge in its interior.

The conditions of Theorem 1 are then valid and we get the error estimate

$$\|\mathbf{u}-\mathbf{u}_h\|_1 + \|p-p_h\|_0 \leq Ch^2(|\mathbf{u}|_3 + |p|_2).$$

We would here like to remark that our analysis shows that the error estimates are valid for an arbitrary mesh $\mathscr{C}_h$. Hence the restriction imposed in References 5 and 6, namely that every $K\in\mathscr{C}_h$ must have at least one vertex in the interior of Ω, is unnecessary.

Example 3[5,8]

$\mathscr{C}_h$ is now defined to be a partitioning of $\bar{\Omega}$ into convex quadrilaterals and the finite element spaces are defined as

$$\mathbf{V}_h = \{\mathbf{v}\in[H_0^1(\Omega)]^2 \mid \mathbf{v}_{|K}\in[Q_2(K)]^2, \quad K\in\mathscr{C}_h\},$$

$$P_h = \{p\in C(\Omega)\cap L_0^2(\Omega) \mid p_{|K}\in Q_1(K), \quad K\in\mathscr{C}_h\}.$$

For this method the macroelement condition is valid for a macroelement consisting of two elements.

To prove this we consider a macroelement $M = K_1 \cup K_2$ and the corresponding reference macroelement $\hat{M} = \hat{K}_1 \cup \hat{K}_2$ as in Figure 3.

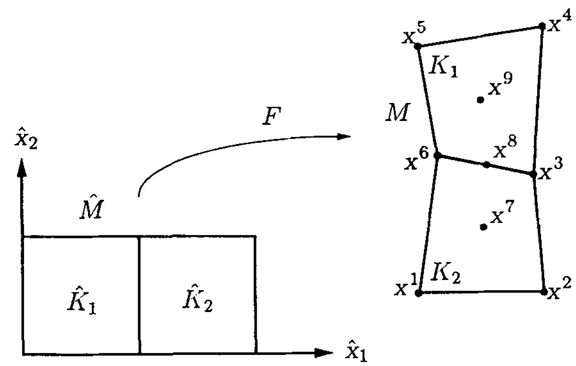

Figure 3

Let $F = (F_1, F_2)$ be the continuous piecewise bilinear mapping from $\hat{M}$ onto M. Defining $\hat{\mathbf{u}}(\hat{x}) = \mathbf{u}(F(\hat{x}))$ and $\hat{p}(\hat{x}) = p(F(\hat{x}))$, we can write

$$(\nabla\cdot\mathbf{u}, p)_M = -(\mathbf{u}, \nabla p)_M = -\sum_{i=1}^{2}\int_{\hat{K}_i} \hat{\mathbf{u}}(\hat{x})^{\mathrm{T}}\,\mathbf{J}_F^{-\mathrm{T}}(\hat{x})\nabla\hat{p}(\hat{x})\,|\mathbf{J}_F(\hat{x})|\,\mathrm{d}\hat{x}$$

for $\mathbf{u}\in\mathbf{V}_{0,M}$ and $p\in P_M$. Here $\mathbf{J}_F$ is the Jacobian matrix of F, $\mathbf{J}_F^{-T}$ is the transpose of $\mathbf{J}_F^{-1}$ and $|\mathbf{J}_F|$ denotes the determinant of $\mathbf{J}_F$. $\hat{\mathbf{u}}(\hat{x})$ and $\nabla\hat{p}(\hat{x})$ are considered as column vectors.

Since

$$|\mathbf{J}_F(\hat{x})|\mathbf{J}_F^{-T}(\hat{x}) = \begin{pmatrix} \partial_2 F_2(\hat{x}) & -\partial_1 F_2(\hat{x}) \\ -\partial_2 F_1(\hat{x}) & \partial_1 F_1(\hat{x}) \end{pmatrix},$$

and F_1 and F_2 are bilinear, we have

$$[|\mathbf{J}_F(\hat{x})|\mathbf{J}_F^{-T}(\hat{x})\nabla\hat{p}(\hat{x})]_{|\hat{K}_i}\in[Q_1(\hat{K}_i)]^2, \quad i = 1, 2.$$

This gives

$$[\hat{\mathbf{u}}(\hat{x})^T \mathbf{J}_F^{-T}(\hat{x})\nabla\hat{p}(\hat{x})|\mathbf{J}_F(\hat{x})|]_{|\hat{K}_i} \in Q_3(\hat{K}_i), \quad i = 1, 2,$$

and hence the composite Simpson rule gives the exact values for the integrals

$$\int_{\hat{K}_i} \hat{\mathbf{u}}(\hat{x})^T \mathbf{J}_F^{-T}(\hat{x})\nabla\hat{p}(\hat{x})|\mathbf{J}_F(\hat{x})|\, d\hat{x}, \quad i = 1, 2.$$

Let us now choose $\mathbf{u} \in \mathbf{V}_{0,M}$ such that the only non-vanishing degrees of freedom are the values of both components at the midpoints x^7 and x^9 of K_1 and K_2 respectively. Then, using the above observation for calculating the integrals, we conclude that the condition $(\nabla \cdot \mathbf{u}, p)_M = 0$ implies that (with $x^i = F(\hat{x}^i)$)

$$\mathbf{J}_F^{-T}(\hat{x}^i)\nabla\hat{p}(\hat{x}^i)\,|\,\mathbf{J}_F(\hat{x}^i)\,| = \mathbf{0}, \quad i = 7, 9.$$

Since $|\mathbf{J}_F(\hat{x}^i)| \neq 0$, $i = 7, 9$, this shows that

$$\nabla\hat{p}(\hat{x}^i) = \mathbf{0}, \quad i = 7, 9.$$

Let now $p_i = \hat{p}(x^i) = \hat{p}(\hat{x}^i)$, $i = 1, 2, \ldots, 6$, be the degrees of freedom for $p \in P_M$. Then the above four conditions for N_M implies that

$$p_1 = p_3 = p_5 = a \quad \text{and} \quad p_2 = p_4 = p_6 = b,$$

where a and b are arbitrary real constants.

Next, when choosing $\mathbf{u} \in \mathbf{V}_{0,M}$ such that the only non-vanishing degrees of freedom are $\mathbf{u}(x^8)$ $(= \hat{\mathbf{u}}(\hat{x}^8))$, we conclude as before that the condition $(\nabla \cdot \mathbf{u}, p)_M = 0$ implies

$$\hat{\mathbf{u}}^T(\hat{x}^8)\{[\mathbf{J}_F^{-T}(\hat{x}^8)\nabla\hat{p}(\hat{x}^8)|\mathbf{J}_F(\hat{x}^8)|]_{|\hat{K}_1} + [\mathbf{J}_F^{-T}(\hat{x}^8)\nabla\hat{p}(\hat{x}^8)|\mathbf{J}_F(\hat{x}^8)|]_{|\hat{K}_2}\} = 0, \tag{23}$$

with the restriction to $\hat{K}_i$, $i = 1, 2$, denoting the limiting value when $\hat{x} \to \hat{x}^8$, $\hat{x} \in \hat{K}_i$. If we now let $\mathbf{u}(x^8) = \hat{\mathbf{u}}(\hat{x}^8) = x^6x^3$, with x^6x^3 denoting the vector from x^6 to x^3, then we have

$$[\mathbf{J}_F^{-1}(\hat{x}^8)\hat{\mathbf{u}}(\hat{x}^8)]_{|\hat{K}_i} = \hat{\mathbf{e}}_2, \quad i = 1, 2,$$

with $\hat{\mathbf{e}}_2 = (0, 1)$. Since $\partial_2\hat{p}$ is continuous at $\hat{x}^8$, (23) reduces to

$$\partial_2\hat{p}(\hat{x}^8)[|\mathbf{J}_F(\hat{x}^8)|_{|\hat{K}_1} + |\mathbf{J}_F(\hat{x}^8)|_{|\hat{K}_2}] = 0.$$

This gives

$$0 = \partial_2\hat{p}(\hat{x}^8) = p_3 - p_6 = a - b,$$

i.e. $a = b$. The macroelement condition is thus proved for a macroelement of two elements.

The partitioning $\mathscr{M}_h$ is then obtained by taking one macroelement for each interior edge of the mesh.

We have thus proved the optimal convergence rate of the method:

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leqslant Ch^2(|\mathbf{u}|_3 + |p|_2).$$

Let us finally remark that the above arguments can be generalized for the whole family of quadrilateral Taylor–Hood methods:

$$\mathbf{V}_h = \{\mathbf{v} \in [H_0^1(\Omega)]^2 \,|\, \mathbf{v}_{|K} \in [Q_k(K)]^2, \quad K \in \mathscr{C}_h\},$$
$$P_h = \{p \in C(\Omega) \cap L_0^2(\Omega) \,|\, p_{|K} \in Q_{k-1}(K), \quad K \in \mathscr{C}_h\},$$

with $k \geqslant 2$, see Reference 9. □

For some further applications of our technique we refer to References 7–9 and 19.

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